

Happy Edges: Threshold-Coloring of Regular Lattices

Md. J. Alam, S. G. Kobourov, S. Pupyrev, and J. Toeniskoetter

Department of Computer Science, University of Arizona, Tucson, USA

Abstract. We study a graph coloring problem motivated by a fun Sudoku-style puzzle. Given a bipartition of the edges of a graph into *near* and *far* sets and an integer threshold t , a *threshold-coloring* of the graph is an assignment of integers to the vertices so that endpoints of near edges differ by t or less, while endpoints of far edges differ by more than t . We study threshold-coloring of tilings of the plane by regular polygons, known as Archimedean lattices, and their duals, the Laves lattices. We prove that some are threshold-colorable with constant number of colors for any edge labeling, some require an unbounded number of colors for specific labelings, and some are not threshold-colorable.

1 Introduction

A Sudoku-style puzzle called *Happy Edges*. Similar to Sudoku, Happy Edges is a grid (represented by vertices and edges), and the task is to fill in the vertices with numbers so as to make all the edges “happy” : a solid edge is happy if the corresponding numbers of its endpoints differ by at most 1, and a dashed edge is happy if the numbers of its endpoints differ by at least 2; see Fig. 1.

In this paper, we study a generalization of the puzzle modeled by a graph coloring problem. The generalization is twofold. Firstly, we consider several underlying regular grids as a base for the puzzle, namely Archimedean and Laves lattices. Secondly, we allow for arbitrary integer difference to distinguish between solid and dashed edges. Thus, the formal model of the puzzle is as follows. The input is a graph with *near* and *far* edges. The goal is to assign integer labels (or colors) to the vertices and compute an integer “threshold” so that the distance between the endpoints of a near edge is within the threshold, while the distance between the endpoints of a far edge is greater than the threshold.

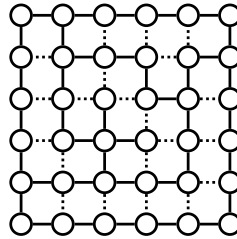


Fig. 1. An example of the *Happy Edges* puzzle: fill in numbers so that nodes separated by a solid edge differ by at most 1 and nodes separated by a dashed edge differ by at least 2. Fearless readers are invited to solve the puzzle before reading further! More puzzles are available online at <http://happy-edges.cs.arizona.edu>.

We consider a natural class of graphs called Archimedean and Laves lattices, which yield symmetric and aesthetically appealing game boards; see Fig. 2. An Archimedean lattice is a graph of an edge-to-edge tiling of the plane using regular polygons with the property that all vertices of the polygons are identical under translation and rotation. Edge-to-edge means that each distinct pair of edges of the tiling intersect at a single endpoint or not at all. There are exactly 11 Archimedean lattices and their dual graphs are the Laves lattices (except for 3 duals which are Archimedean). We are interested in identifying the lattices that can be appropriately colored for any prescribed partitioning of edges into near and far. Such lattices can be safely utilized for the *Happy Edges* puzzle, as even the simplest random strategy may serve as a puzzle generator.

Another motivation for studying the threshold coloring problem comes from the geometric problem of *unit-cube proper contact representation* of planar graphs. In such a representation, vertices are represented by unit-size cubes, and edges are represented by common boundary of non-zero area between the two corresponding cubes. Finding classes of planar graphs with unit-cube proper contact representation was posed as an open question by Bremner *et al.* [4]. As shown in [1], threshold-coloring can be used to find such a representation of certain graphs.

Terminology and Problem Definition: An *edge labeling* of a graph $G = (V, E)$ is a map $l : E \rightarrow \{N, F\}$. If $(u, v) \in E$, then (u, v) is called *near* if $l(u, v) = N$ and u is said to be *near* to v . Otherwise, (u, v) is called *far* and u is *far* from v . A *threshold-coloring* of G with respect to l is a map $c : V \rightarrow \mathbb{Z}$ such that there exists an integer $t \geq 0$, called the *threshold*, satisfying for every edge $(u, v) \in E$, $|c(u) - c(v)| \leq t$ if and only if $l(u, v) = N$. If m is the minimum value of c , and M the maximum, then $r = M - m$ is the *range* of c . The map c is called a (r, t) -*threshold-coloring* and G is *threshold-colorable* or (r, t) -*threshold-colorable* with respect to l .

If G is (r, t) -threshold-colorable with respect to every edge labeling, then G is (r, t) -*total-threshold-colorable*, or simply *total-threshold-colorable*. If G is not (r, t) -total-threshold-colorable, then G is *non- (r, t) -total-threshold-colorable*, or *non-total-threshold-colorable* if G is non- (r, t) -total-threshold-colorable for all values of (r, t) .

In an edge-to-edge tiling of the plane by regular polygons, the *species* of a vertex v is the sequence of degrees of polygons that v belongs to, written in clockwise order. For example, each vertex of the triangle lattice has 6 triangles, and so has species $(3, 3, 3, 3, 3, 3)$. A vertex of the square lattice has species $(4, 4, 4, 4)$, and vertices of the octagon-square lattice have species $(4, 8, 8)$. Exponents are used to abbreviate this: $(4, 8^2) = (4, 8, 8)$. The *Archimedean tilings* are the 11 tilings by regular polygons such that each vertex has the same species; we use this species to refer to the lattice. For example, (6^3) is the hexagon lattice, and $(3, 12^2)$ is the lattice with triangles and dodecagons. An *Archimedean lattice* is an infinite graph defined by the edges and vertices of an Archimedean tiling. If A is an Archimedean lattice, then we refer to its dual graph as $D(A)$. The lattice (3^6) of triangles and the lattice (6^3) of hexagons are dual to each other, whereas the lattice (4^4) of squares is dual to itself. The duals of the other 8 Archimedean lattices are not Archimedean, and these are referred to as *Laves lattices*; see Fig. 2. By an abuse of notation, any induced subgraph of an Archimedean or Laves lattice is called an Archimedean or Laves lattice.

$(4,6,12)$ Yes: $r = 9, t = 2$	$(3,12^2)$ Yes: $r = 9, t = 2$	$(3,4,6,4)$ Maybe	(3^6) No
$(3^3, 4^2)$ No	$(3,6,3,6)$ Maybe	(6^3) Yes: $r = 5, t = 1$	$(4, 8^2)$ Yes: $r = 5, t = 1$
(4^4) Maybe: $r = \mathcal{O}(V)$	$(3^4, 6)$ No	$(3^2, 4, 3, 4)$ No	$D(3^2, 4, 3, 4)$ Yes: $r, t = \mathcal{O}(V)$
$D(3, 12^2)$ No	$D(4, 6, 12)$ No	$D(4, 8^2)$ No	$D(3,4,6,4)$ Maybe: $r = \mathcal{O}(V)$
$D(3, 6, 3, 6)$ Maybe: $r = \mathcal{O}(V)$	$D(3^4, 6)$ Yes: $r, t = \mathcal{O}(V)$	$D(3^3, 4^2)$ Maybe	

Fig. 2. The 11 Archimedean and 8 Laves lattices. With each lattice's name, we provide a summary of results concerning the threshold-coloring of the lattice. For those which are total-threshold-colorable we list the best known values of r and t . For those which might be total-threshold-colorable, we list known constraints on r and t .

Related Work: Many problems in graph theory deal with coloring or labeling the vertices of a graph [13] and many graph classes are defined based on such a coloring [3]. Alam *et al.* [1] introduce threshold-coloring and show that deciding whether a graph is threshold colorable with respect to an edge labeling is equivalent to the graph sandwich problem for proper-interval-representability, which is NP-complete [9]. They also

show that graphs with girth (that is, length of shortest cycle) at least 10 are always total-threshold-colorable.

Total-threshold-colorable graphs are related to threshold and difference graphs. In *threshold graphs* there exists a real number S and for every vertex v there is a real weight a_v so that (v, w) is an edge if and only if $a_v + a_w \geq S$ [12]. A graph is a *difference graph* if there is a real number S and for every vertex v there is a real weight a_v so that $|a_v| < S$ and (v, w) is an edge if and only if $|a_v - a_w| \geq S$ [11]. Note that for both these classes the existence of an edge is determined wholly by the threshold S , while in our setting the edges defined by the threshold must also belong to the original (not necessarily complete) graph.

Threshold-colorability is related to the *integer distance graph* representation [5,6]. An integer distance graph is a graph with the set of integers as vertex set and with an edge joining two vertices u and v if and only if $|u - v| \in D$, where D is a subset of the positive integers. Clearly, an integer distance graph is threshold-colorable if the set D is a set of consecutive integers. Also related is *distance constrained graph labeling*, denoted by $L(p_1, \dots, p_k)$ -labeling, a labeling of the vertices of a graph so that for every pair of vertices with distance at most $i \leq k$ the difference of their labels is at least p_i . $L(2, 1)$ -labelings are well-studied [8] and minimizing the number of labels is NP-complete, even for diameter-2 graphs [10]. It is NP-complete to determine if a labeling exists with at most k labels for every fixed integer $k \geq 4$ [7].

Our Results: We study the threshold-colorability of the Archimedean and Laves lattices; see Fig. 2 for an overview of the results. First, we prove that 6 of them are threshold-colorable for any edge labeling. Hence, the *Happy Edges* puzzle always have a solution on these lattices. Then we show that 7 of the lattices have an edge labeling admitting no threshold-coloring. Finally, for 3 no constant range of colors suffices. The puzzle motivating the problem is available at <http://happy-edges.cs.arizona.edu>; see also Fig. 13.

2 Total-Threshold-Colorable Lattices

Given a graph $G = (V, E)$, a subset I of V is called *2-independent* if the shortest path between any two distinct vertices of I has length at least 3. For a subset V' of V , we denote the subgraph of G induced by V' as $G[V']$. We give an algorithm for threshold-coloring graphs whose vertex set has a partition into a 2-independent set I and a set T such that $G[T]$ is a forest. Dividing G into a forest and 2-independent set has been used for other graph coloring problems, for example in [2,14] for the star coloring problem.

2.1 The (6^3) and $(4, 8^2)$ Lattices

Lemma 1. *Suppose $G = (I \cup T, E)$ is a graph such that I is 2-independent, $G[T]$ is a forest, and I and T are disjoint. Then G is $(5, 1)$ -total-threshold-colorable.*

Proof. Suppose $l : E \rightarrow \{N, F\}$ is an edge labeling. For each $v \in I$, set $c(v) = 0$. Each vertex in T is assigned a color from $\{-2, -1, 1, 2\}$ as follows. Choose a component T' of $G[T]$, and select a root vertex w of T' . If w is far from a neighbour in I , set $c(w) = 2$.

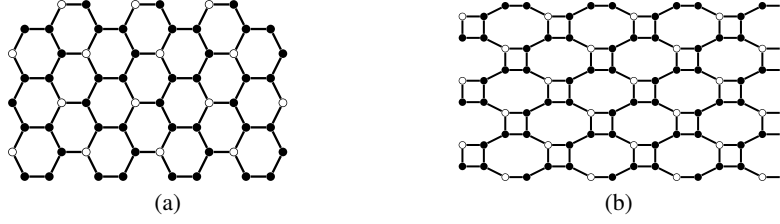


Fig. 3. Decomposing vertices into a 2-independent set, shown in white, and a forest. (a) The (6^3) lattice. (b) The $(4, 8^2)$ lattice.

Otherwise, $c(w) = 1$. Now we conduct breadth first search on T' , coloring each vertex as it is traversed. When we traverse to a vertex $u \neq w$, it has one neighbour $x \in T'$ which has been colored, and at most one neighbour $v \in I$. If v exists, we choose the color $c(u) = 1$ if $l(u, v) = N$, and $c(u) = 2$ otherwise. Then, if the edge (u, x) is not satisfied, we multiply $c(u)$ by -1 . If v does not exist, we choose $c(u) = 1$ or -1 to satisfy the edge (u, x) . By repeating the procedure on each component of $G[T]$, we construct a $(5, 1)$ -threshold-coloring of G with respect to the labeling l . \square

The (6^3) and $(4, 8^2)$ lattices have such a decomposition; see Fig. 3. Hence,

Theorem 1. *The (6^3) and $(4, 8^2)$ lattices are $(5, 1)$ -total-threshold-colorable.*

2.2 The $(3, 12^2)$ and $(4, 6, 12)$ Lattices

In order to color the lattices, we use $(9, 2)$ -color space, that is, threshold 2 and 9 colors, such as $\{0, \pm 1, \pm 2, \pm 3, \pm 4\}$. This color-space has the following properties.

Lemma 2. *Consider a path with 3 vertices (v_0, v_1, v_2) , such that v_0, v_2 have colors $c(v_0), c(v_2)$ in $\{0, \pm 1, \pm 2, \pm 3, \pm 4\}$. For threshold 2 and any edge labeling,*

- (a) *If $c(v_0) = 0$, and $c(v_2) \in \{\pm 1, \pm 2, \pm 3, \pm 4\}$, then we can choose $c(v_1)$ in $\{\pm 2, \pm 3\}$.*
- (b) *If $c(v_0) = 0$ and $c(v_2) \in \{\pm 2, \pm 3, \pm 4\}$, then we can choose $c(v_1)$ in $\{\pm 2, \pm 4\}$.*
- (c) *If $c(v_0) = \pm 1$, and $c(v_2) \in \{\pm 2, \pm 3\}$, then we can choose $c(v_1)$ in $\{\pm 1, \pm 4\}$.*

Proof. (a) First, we choose $c(v_1) = \pm 2$ if v_1 is near to v_0 , and ± 3 otherwise. Then, if v_1 is near to v_2 , choose the sign of $c(v_1)$ to agree with $c(v_2)$. Otherwise choose the sign of $c(v_1)$ to be opposite $c(v_2)$.

(b) Choose $c(v_1) = \pm 2$ if v_1 is near to v_0 , and ± 4 otherwise. Then, choose the sign of $c(v_1)$ as before.

(c) Choose $c(v_1) = \pm 1$ if v_1 is near to v_0 , and $c(v_1) = \pm 4$ otherwise. Then, choose the sign of $c(v_1)$ as before. \square

On a high level, our algorithms for the $(3, 12^2)$ and $(4, 6, 12)$ lattices are very similar to each other: we identify small “patches”, and then assemble them into the lattice; see Figs. 4-5. We first show how to color a patch for $(3, 12^2)$ and then for $(4, 6, 12)$.

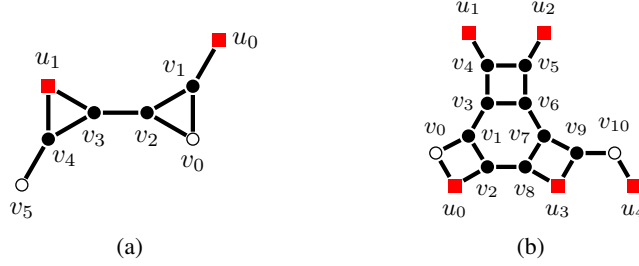


Fig. 4. Illustration of Lemma 3 and 4. (a) A subgraph of the $(3, 12^2)$ lattice. (b) A subgraph of the $(4, 6, 12)$ lattice. Square vertices are labeled 0.

Lemma 3. *Let G be the graph shown in Fig 4(a). Suppose $c(u_0) = c(u_1) = 0$ and $c(v_0) = \pm 1$. Then for any edge labeling, this coloring can be extended to a $(9, 2)$ -threshold-coloring of G such that v_5 is colored 1 or -1 .*

Proof. Assume $c(v_0) = 1$. We apply Lemma 2(a) to the path (u_0, v_1, v_0) to choose a color for v_1 in $\{\pm 2, \pm 3\}$, then apply part (c) of the lemma to the path (v_0, v_2, v_1) to choose $c(v_2) \in \{\pm 1, \pm 4\}$. Then $c(v_3)$ is chosen in $\{\pm 2, \pm 3\}$ using part (a) of the lemma on the path (u_1, v_3, v_2) , and finally $c(v_4) \in \{\pm 2, \pm 3\}$ is chosen using part (a) on the path (u_1, v_4, v_3) . Then we may choose $c(v_5) = 1$ or -1 so that it is near or far from $c(v_4)$. \square

A similar lemma concerns the $(4, 6, 12)$ lattice; see the proof in the Appendix.

Lemma 4. *Let G be the graph shown in Fig. 4(b), and consider any edge labeling. Suppose that $c(u_i) = 0$, for $i = 0, \dots, 4$, and $c(v_0)$ is a fixed color in $\{\pm 2, \pm 4\}$ that satisfies the label of (v_0, u_0) . Then we can extend this partial coloring to a coloring c of all of G , so that c is a $(9, 2)$ -threshold-coloring of G with respect to the edge labeling, and $c(v_{10})$ is in $\{\pm 2, \pm 4\}$.*

Theorem 2. *The $(3, 12^2)$ and $(4, 6, 12)$ lattices are $(9, 2)$ -total-threshold-colorable.*

Proof. We prove the claim for $(3, 12^2)$; see Appendix for the $(4, 6, 12)$ proof.

First, we join several copies of the graph G in Lemma 3. Let G_1, \dots, G_n be copies of G . Let us call $u_{i,k}$ and $v_{j,k}$ the vertices in G_k , corresponding to u_i, v_j ($i = 0$ or $1, 0 \leq j \leq 5$). For $1 \leq k < n$, we set $v_{5,k} = v_{0,k+1}$. This defines a single row of the $(3, 12^2)$ lattice. We can construct a $(9, 2)$ -threshold-coloring of this chain of G_1, \dots, G_n by giving the vertex $v_{0,1}$ the color 1 and repeatedly applying Lemma 3.

To construct the next row, we add a copy of G connected to G_i and G_{i+2} for each odd i with $1 \leq i \leq k - 2$, by identifying $u_{1,i} = u_0$ and $u_{0,i+2} = u_1$. We then join the copies of G added above the first row in the same way that the copies G_1, \dots, G_n were joined. By repeatedly adding new rows, we complete the construction of the $(3, 12^2)$ lattice. We can threshold-color each row, and since the rows are connected only by vertices colored 0, the entire graph is $(9, 2)$ -total-threshold-colorable; see Fig. 5. \square

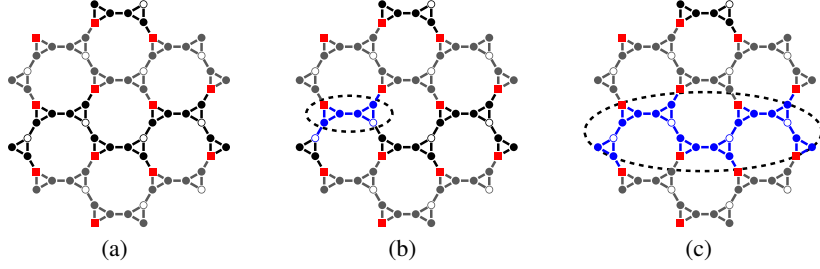


Fig. 5. Threshold-coloring the $(3, 12^2)$ lattice. (a) Identifying the rows separated by square vertices. (b) One patch has been colored, shown inside the oval. (c) Coloring an entire row.

2.3 The $D(3^2, 4, 3, 4)$ and $D(3^4, 6)$ Lattices

Here we give an algorithm for threshold-coloring of the $D(3^2, 4, 3, 4)$ and $D(3^4, 6)$ lattices using $\mathcal{O}(|V|)$ colors and $\mathcal{O}(|V|)$ threshold. By k -vertex, we mean a vertex of degree k . We use the following strategy. First, we construct an independent set I . For the $D(3^2, 4, 3, 4)$ lattice, I consists of all the 4-vertices; see Fig. 6(b). For the $D(3^4, 6)$ lattice, I consists of all the 6-vertices and some 3-vertices; see Fig. 11(a). Consider an edge labeling $l : E \rightarrow \{N, F\}$. We color all the vertices of I using $|I|$ different colors such that each of these vertices gets a unique color. Next we color the remaining 3-vertices so that for each edge $e = (u, v)$ of the graph $|c(u) - c(v)| \leq |I|$ if and only if $l(e) = N$. By definition, this gives a threshold-coloring of the graph with threshold $|I|$. Note that for both these lattices, the 3-vertices remaining after the vertices in I are removed induce a matching, that is, a set of edges with disjoint end-vertices. We color these 3-vertices in pairs, defined by the matching.

We now describe the algorithm. Consider the graph G_6 with edges e_0, \dots, e_4 partitioned into near and far and coloring $c : \{w_1, w_2, w_3, w_4\} \rightarrow \{k + 2, \dots, 2k + 1\}$ for some integer $k > 0$ such that each of the vertices gets a unique color; see Fig. 6(a).

After possible renaming assume that if $l(e_1) \neq l(e_2)$ then $l(e_1) = N, l(e_2) = F$, and if $l(e_3) \neq l(e_4)$ then $l(e_3) = N, l(e_4) = F$. We say that c is *extendible* with respect to l if at least one of the following conditions hold.

1. $l(e_1) = l(e_2)$ or $l(e_3) = l(e_4)$; that is, at least one pair between $\{e_1, e_2\}$ and $\{e_3, e_4\}$ gets the same edge labeling from l .
2. $l(e_0) = N$ and $c(w_1) < c(w_2)$ if and only if $c(w_3) < c(w_4)$.
3. $l(e_0) = F$ and $c(w_1) < c(w_2)$ if and only if $c(w_3) > c(w_4)$.

The following lemma proves that if c is extendible with respect to l , then there is a $(3k + 1, k)$ -threshold-coloring of G_6 ; see Appendix for the proof.

Lemma 5. *Consider the graph G_6 in Fig. 6(a). Let $l : E \rightarrow \{N, F\}$ be an edge labeling of E and let $c : (V - \{u, v\}) \rightarrow \{k + 2, \dots, 2k + 1\}$ be an extendible coloring with respect to l . Then there exist colors $c(u)$ and $c(v)$ for u and v from the set $\{1, \dots, 3k + 2\}$ such that c is a threshold-coloring of G for l with threshold k .*

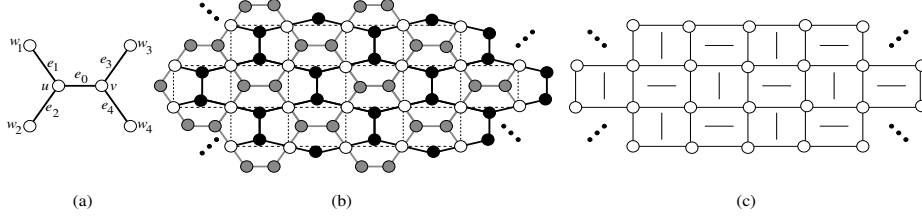


Fig. 6. (a) The graph G_6 , (b)–(c) Illustration for the proof of Theorem 3.

Theorem 3. *The $D(3^2, 4, 3, 4)$ lattice is $(3m+2, m)$ -total-threshold-colorable with m equal to the number of 4-vertices in the lattice.*

Proof. Let G be a subgraph of $D(3^2, 4, 3, 4)$ and let l be an edge labeling of G . Let m be the number of 4-vertices in G . Assign the threshold $t = m$. The remaining vertices V_2 of G have degree 3 and they form a matching. Each edge (u, v) between these vertices is surrounded by exactly four 4-vertices, which are the other neighbors of u and v ; see Fig. 6(b). Call this edge *horizontal* if it is drawn horizontally in Fig. 6(b); otherwise call it *vertical*. Our goal is to color the vertices of V_1 so that for each horizontal and vertical edge of G , this coloring is extendible with respect to l .

Consider only the 4-vertices V_1 of G and add an edge between two of them if they have a common neighbour in G . This gives a square grid H ; see Fig. 6(c). Each square S of H is *horizontal* (*vertical*) if it is associated with a horizontal (*vertical*) edge in G . Let u_1, u_2, u_3 and u_4 be the left-top, right-top, left-bottom and right-bottom vertices of S and let c_1, c_2, c_3 and c_4 be the colors assigned to them. Suppose S is a vertical square. Then in order to make the coloring extendible with respect to l , we need that $c_1 < c_2$ or $c_1 > c_2$ implies exactly one of the two relations $c_3 < c_4$ and $c_3 > c_4$, depending on the edge-label of the associated vertical edge. Similarly if S is a horizontal square then the relation between c_1 and c_3 implies a relation between c_2 and c_4 depending on the edge label of the associated horizontal edge. Consider that an edge in H is directed from the vertex with the smaller color to the vertex with the larger color. Then for the coloring to be extendible to l , we need that for a vertical square S the direction of the edge (u_3, u_4) is the same as or opposite to that of (u_1, u_2) and for a horizontal edge the direction of (u_2, u_4) is the same as or opposite to that of (u_1, u_3) , depending on the edge-label of the associated vertical or horizontal edge. We call this a *constraint* defined on S . We now show how to find an acyclic orientation of H so that the constraints defined on the squares are satisfied.

We traverse the square grid H from left-top to right-bottom. We thus assume that when we are traversing a particular square S , the orientations of its top and left edge have already been assigned. We now orient the bottom and right edge so that the constraint defined on S is satisfied. We also maintain an additional invariant that the right-bottom vertex of each square is either a source or a sink; that is, the incident edges are either both outgoing or both incoming. Consider the traversal of a particular square S . If S is vertical, then the direction of the bottom edge is defined by the direction of the top-edge and the constraint for S . We then orient the right edge so that the right-bottom vertex is either a source or a sink; that is, we orient the right edge upward (downward

resp.) if the bottom edge is directed to the left (right resp.). Similarly if S is horizontal, the direction of the right edge is defined by the constraint and we give direction to the bottom edge so that the right-bottom vertex is either a source or a sink. We thus have an orientation of the edges of H satisfying all the constraints at the end of the traversal. It is easy to see that this orientation defines a directed acyclic graph. For a contradiction assume that there is a directed cycle C in H . Then take the bottommost vertex x of C which is to the right of every other bottommost vertex. Then x is either a source or a sink by our orientation and hence cannot be part of a directed cycle, a contradiction.

Once we have the directed acyclic orientation of H , we compute the coloring $c : V_1 \rightarrow \{1, \dots, m\}$ of the vertices V_1 of G in a topological sort of this directed acyclic graph. We shift this color-space to $\{m+2, \dots, 2m+1\}$ by adding $m+1$ to each color. This coloring is extendible to the edge labeling l since the orientation satisfies all the constraints. Thus by Lemma 5 we can color all the 3-vertices of G , taking $k = m$. We thus have a threshold-coloring of G with $3m+2$ colors and a threshold m . \square

A similar result holds for the $D(3^4, 6)$ lattice; see Appendix for the proof.

Theorem 4. *The $D(3^4, 6)$ lattice is total-threshold-colorable with $\mathcal{O}(|V|)$ threshold and $\mathcal{O}(|V|)$ colors, where V is the vertex set.*

3 Non-Total-Threshold-Colorable Lattices

In this section, we consider several lattices that cannot be threshold-colored. We begin with a useful lemma.

Lemma 6. *Consider a K_3 defined on $\{v_0, v_1, v_2\}$ and a 4-cycle $(u_0, u_1, u_2, u_3, u_0)$. Then for a given threshold t , a threshold-coloring c and edge labeling l :*

- (a) *Let $l(v_0, v_2) = F$ and $l(v_0, v_1) = l(v_1, v_2) = N$. If $c(v_0) < c(v_1)$, then $c(v_1) < c(v_2)$.*
- (b) *Let $l(v_0, v_2) = N$ and $l(v_0, v_1) = l(v_1, v_2) = F$. If $c(v_0) < c(v_1)$, then $c(v_2) < c(v_1)$.*
- (c) *Let $l(u_0, u_3) = l(u_2, u_3) = F$ and $l(u_0, u_1) = l(u_1, u_2) = N$. If $c(u_0) < c(u_3)$, then $c(u_1) < c(u_3)$ and $c(u_2) < c(u_3)$.*
- (d) *Let $l(u_0, u_1) = l(u_2, u_3) = F$ and $l(u_0, u_3) = l(u_1, u_2) = N$. If $c(u_0) < c(u_1)$, then $c(u_0) < c(u_2)$, $c(u_3) < c(u_1)$, and $c(u_3) < c(u_2)$.*

Note that we can replace $<$ with $>$ in each case.

Proof. (a) Suppose that $c(v_0) < c(v_1)$. Then $c(v_1) - t \leq c(v_0) < c(v_1)$. If $c(v_2) < c(v_1)$, then also $c(v_1) - t \leq c(v_2) < c(v_1)$, but then $|c(v_0) - c(v_2)| \leq t$, a contradiction. Thus $c(v_1) < c(v_2)$.

(b) Suppose that $c(v_0) < c(v_1)$. If $c(v_2) > c(v_1)$, then $c(v_0) < c(v_1) < c(v_2)$ and $|c(v_0) - c(v_2)| \leq t$, so $|c(v_0) - c(v_1)| \leq t$, a contradiction. Hence, $c(v_2) < c(v_1)$.

(c) Suppose that $c(u_0) < c(u_3)$. Then $c(u_0) < c(u_3) - t$ and $|c(u_0) - c(u_1)| \leq t$, so $c(u_1) < c(u_3)$, and therefore, $c(u_2) < c(u_3) + t$, so $c(u_2)$ must be less than $c(u_3)$ since $|c(u_2) - c(u_3)| > t$.

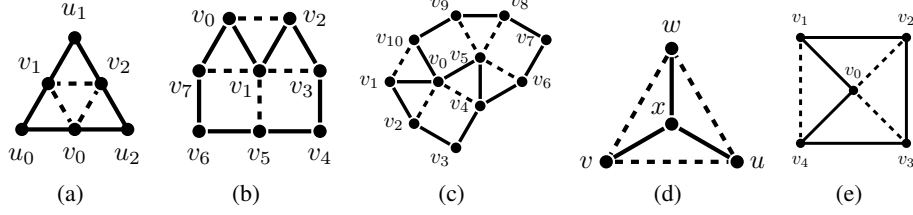


Fig. 7. Non-total-threshold-colorable graphs with dashed edges labeled F and solid ones labeled N . (a) A subgraph of (3^6) and $(3^4, 6)$. (b) A subgraph of $(3^3, 4^2)$. (c) A subgraph of $(3^2, 4, 3, 4)$. (d) A subgraph of $D(3, 12^2)$. (e) A subgraph of $D(4, 6, 12)$ and $D(4, 8^2)$.

(d) Suppose that $c(u_0) < c(u_1)$. Then $c(u_0) < c(u_1) - t$, $c(u_2) \geq c(u_1) - t$, and so $c(u_1) < c(u_2)$. $c(u_3) < c(u_1)$ since $|c(u_0) - c(u_3)| \leq t$. If $c(u_3) > c(u_2)$, then $c(u_1) - t \leq c(u_2) < c(u_3) < c(u_1)$, so $|c(u_2) - c(u_3)| \leq t$, a contradiction. \square

Theorem 5. The (3^6) , $(3^4, 6)$, $(3^3, 4^2)$, $(3^2, 4, 3, 4)$, $D(3, 12^2)$, $D(4, 6, 12)$, and $D(4, 8^2)$ lattices are non-total-threshold-colorable.

Proof. It is easy to see that a cycle with exactly 1 far edge is not $(r, 0)$ -threshold-colorable, so we need only prove the lattices are not (r, t) -total-threshold-colorable for $t > 0$. In this proof we assume that r is an arbitrary integer and $t > 0$.

The (3^6) and $(3^4, 6)$ lattices contain the subgraph G in Fig. 7(a). Suppose there exists an (r, t) -threshold-coloring c . Without loss of generality we may assume that $c(v_0) < c(v_1) < c(v_2)$. Then $c(v_0) + t < c(v_1)$ and $c(v_1) + t < c(v_2)$, so $c(v_0) + 2t < c(v_2)$. Since the edges (v_0, u_2) and (v_2, u_2) are labeled N , we have $|c(v_2) - c(v_0)| < |c(v_2) - c(u_2)| + |c(v_0) - c(u_2)| \leq 2t$, which is a contradiction.

A subgraph of $(3^3, 4^2)$ is shown in Fig. 7(b). If c is an (r, t) -threshold-coloring and w.l.o.g. $c(v_0) < c(v_1) < c(v_2)$, then we repeatedly apply Lemma 6 to the vertices around the boundary. First we obtain $c(v_2) < c(v_3)$, and since $c(v_1) < c(v_3)$ we get $c(v_4)$ and $c(v_5)$ larger than $c(v_1)$, which leads to $c(v_6)$ and $c(v_7)$ greater than $c(v_1)$. Then we must have $c(v_1) < c(v_0) < c(v_7)$, which means both $c(v_0)$ and $c(v_2)$ are in the set $\{c(v_1), c(v_1) - 1, \dots, c(v_1) - t\}$, contradicting the fact that the edge (v_0, v_2) is labeled far.

For the $(3^2, 3, 4, 3)$ lattice, consider the graph in Fig. 7(c). Suppose there exists an (r, t) -threshold-coloring c . Assume w.l.o.g. that $c(v_0) = 0 < c(v_1)$. By Lemma 6, $c(v_2)$, $c(v_3)$, and $c(v_4)$ are positive. Additionally, $c(v_0) < c(v_5) < c(v_4) < c(v_6)$, and $c(v_7)$, $c(v_8)$, $c(v_9)$ must all be greater than $c(v_5)$. Since $c(v_5) > 0$, we have $c(v_9) \geq t + 1$, and since the edge (v_9, v_{10}) is labeled N it must be that $c(v_{10}) > 0$. By Lemma 6(a), we have $c(v_{10}) < c(v_0) < c(v_1)$, a contradiction.

$D(3, 12^2)$ contains K_4 as a subgraph. Label the edges of K_4 so that each edge on the outer face is far, and the other edges are near as in Fig. 7(d). Let u, v, w be the vertices of the outerface, x be the interior vertex, and assume an (r, t) -threshold-coloring c exists. Assume that $c(u) < c(x)$. From Lemma 6(a), we then get that $c(x) < c(v)$, which implies by the same lemma that $c(w) < c(x)$, and thus $c(x) < c(u)$, a contradiction.

$D(4, 6, 12)$, and $D(4, 8^2)$ contains the subgraph in Fig. 7(e). Assume an (r, t) -threshold-coloring c exists. Then without loss of generality say $c(v_4) < c(v_0) < c(v_1)$. By Lemma 6(a) it follows that $c(v_1) < c(v_2)$ so $c(v_2) > c(v_0)$. By Lemma 6(b) we have $c(v_3) > c(v_0)$ and thus $c(v_4) > c(v_0)$, a contradiction. \square

4 Graphs With Unbounded Colors

We consider lattices, which are not (r, t) -total-threshold-colorable for any fixed $r > 0$.

Theorem 6. *For every $r > 0$, there exists finite subgraphs of (4^4) , $D(3, 4, 6, 4)$, and $D(3, 6, 3, 6)$, which are not (r, t) -total-threshold-colorable for any $t \geq 0$.*

Proof. We prove the claim for the (4^4) lattice (square grid); see Appendix for the rest.

By the comment in the proof of Theorem 5, we know that the (4^4) lattice is not $(r, 0)$ -total-threshold-colorable for any r . Let S be the infinite square grid, drawn as in Fig. 8. A vertex v in S has north, east, south, and west neighbors. If $P = (v_1, \dots, v_j)$ is a path in S , we call P a *north path* if v_{i+1} is the north neighbour of v_i for all $1 \leq i < j$. *East*, *south*, and *west paths* are defined similarly and these paths are uniquely defined for a given start v_i and number of vertices j .

For each odd $n > 0$, we define a path $S_n = (v_1, \dots, v_{n^2})$ in S . Let S_1 be the path consisting of a single chosen vertex v_1 of S . Let $k = n + 2$, and recursively construct S_k from S_n by first adding the east neighbour v_{n^2+1} of v_{n^2} to S_n . Then, we add the north path $(v_{n^2+1}, \dots, v_{n^2+k})$, the west path $(v_{n^2+k}, \dots, v_{n^2+2k})$, the south path $(v_{n^2+2k}, \dots, v_{n^2+3k})$, and the east path $(v_{n^2+3k}, \dots, v_{n^2+4k})$; see Fig. 8.

With S_n defined for every odd n , let $G_n = (V_n, E_n)$ be the subgraph of S induced by the vertices of S_n , and let $l_n : E_n \rightarrow \{N, F\}$ be an edge labeling such that $l_n(e) = N$ if and only if e is in S_n . The graph G_7 is shown in Fig. 8. We now prove that G_n requires at least n colors to threshold-color, for any threshold $t > 0$. W.l.o.g. suppose that c is a threshold coloring such that $c(v_4) > c(v_1)$. Note that the cycles (v_4, v_5, v_6, v_1) , (v_6, v_7, v_8, v_1) , and (v_8, v_9, v_2, v_1) match the cycles in Lemma 6, implying that $c(v_6), c(v_8)$ and $c(v_9)$ are greater than $c(v_1)$. This serves as the basis for induction. Suppose that for some odd $k > 1$, the vertex $c(v_{k^2}) > c(v_{(k-2)^2})$ for any assignment c of colors to the vertices of G_n , so long as $c(v_4) > c(v_1)$ and c is an (r, t) -threshold-coloring for some $r > 0$. Then we consider the color $c(v_i)$, for $k^2 < i \leq (k+2)^2$. There are three cases. In the first, v_i is the interior vertex of a north, east, west, or south path in S_{k+2} . Then v_i is on a cycle $(v_{i-1}, v_i, v_j, v_{j-1})$, $j \leq k^2$, with $l(v_i, v_{i-1}) = l(v_j, v_{j-1}) = N$ and $l(v_i, v_j) = l(v_{i-1}, v_{j-1}) = F$. By Lemma 6, we have $c(v_i) > c(v_j)$ and $c(v_i) > c(v_{j-1})$ so long as $c(v_{i-1}) > c(v_{j-1})$. In the second case, v_i is part of a 4 cycle $(v_{i-1}, v_i, v_{i+1}, v_j)$, $j \leq k^2$, with $l(v_{i-1}, v_i) = l(v_i, v_{i+1}) = N$, and the other edges labeled F . Again by Lemma 6, we have $c(v_i) > c(v_j)$ and

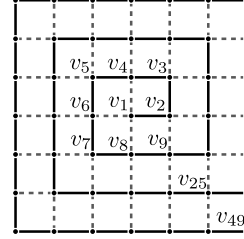


Fig. 8. An example of a square lattice requiring an arbitrary number of colors. Dashed edges are far.

$c(v_{i+1}) > c(v_j)$ so long as $c(v_{i-1}) > c(v_j)$. The third case is the same, except v_i is in the place of v_{i+1} .

Given these three cases and the assumption that $c(v_{k^2}) > c(v_{(k-2)^2})$, we conclude that $c(v_{(k+2)^2}) > c(v_{k^2})$ for each odd $k > 1$. Therefore, the graph G_n , with edge labeling l_n , requires a distinct color for each of $c(v_1), c(v_{3^2}), \dots, c(v_{n^2})$. \square

5 Conclusion and Open Questions

Motivated by a fun Sudoku-style puzzle, we considered the threshold-coloring problem for Archimedean and Laves lattices. For some of these lattices, we presented new coloring algorithms, while for others we found subgraphs that cannot be threshold-colored. Several challenging open questions remain. While we showed that subgraphs of the square lattice and two others require unbounded number of colors, we do not know whether finite subgraphs thereof are threshold-colorable. In the context of the puzzle, it would be useful to find algorithms for checking threshold-colorability for a particular subgraph of a lattice, rather than checking all subgraphs, as required in total-threshold-colorability. There are other interesting variants of the problem pertinent to the puzzle. One restricts the problem by allowing only a fixed number of colors to assign to the vertices. Another fixes the colors of certain vertices, similar to fixing boxes in Sudoku.

References

1. M. J. Alam, S. Chaplick, G. Fijavz, M. Kaufmann, S. G. Kobourov, and S. Pupyrev. Threshold coloring and unit-cube contact representation of graphs. In *Graph-Theoretic Concepts in Computer Science*, pages 26–37, 2013.
2. M. O. Albertson, G. G. Chappell, H. A. Kierstead, A. Kündgen, and R. Ramamurthi. Coloring with no 2-colored P_4 . *Electron. J. Combin.*, 11(1):R26, 2004.
3. A. Brandstädt, V. B. Le, and J. P. Spinrad. *Graph classes: a survey*. Society for Industrial and Applied Mathematics, 1999.
4. D. Bremner, W. Evans, F. Frati, L. Heyer, S. Kobourov, W. Lenhart, G. Liotta, D. Rappaport, and S. Whitesides. On representing graphs by touching cuboids. In *Graph Drawing*, pages 187–198, 2012.
5. R. Eggleton, P. Erdős, and D. Skilton. Colouring the real line. *Journal of Combinatorial Theory, Series B*, 39(1):86–100, 1985.
6. M. Ferrara, Y. Kohayakawa, and V. Rödl. Distance graphs on the integers. *Combinatorics Probability and Computing*, 14(1):107–131, 2005.
7. J. Fiala, T. Kloks, and J. Kratochvíl. Fixed-parameter complexity of λ -labelings. *Discrete Applied Mathematics*, 113(1):59–72, 2001.
8. J. Fiala, J. Kratochvíl, and A. Proskurowski. Systems of distant representatives. *Discrete Applied Mathematics*, 145(2):306–316, 2005.
9. M. C. Golumbic, H. Kaplan, and R. Shamir. Graph sandwich problems. *Journal of Algorithms*, 19(3):449–473, 1995.
10. J. R. Griggs and R. K. Yeh. Labelling graphs with a condition at distance 2. *SIAM Journal on Discrete Mathematics*, 5(4):586–595, 1992.
11. P. L. Hammer, U. N. Peled, and X. Sun. Difference graphs. *Discrete Applied Mathematics*, 28(1):35–44, 1990.
12. N. V. Mahadev and U. N. Peled. *Threshold Graphs and Related Topics*. North Holland, 1995.

13. F. Roberts. From garbage to rainbows: Generalizations of graph coloring and their applications. *Graph Theory, Combinatorics, and Applications*, 2:1031–1052, 1991.
14. C. Timmons. Star coloring high girth planar graphs. *The Electronic Journal of Combinatorics*, 15(1):R124, 2008.

Appendix

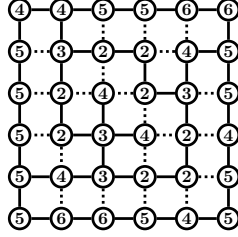


Fig. 9. A solution to the puzzle given in Fig. 1.

Here we provide detailed proofs of theorems omitted from the body of the paper.

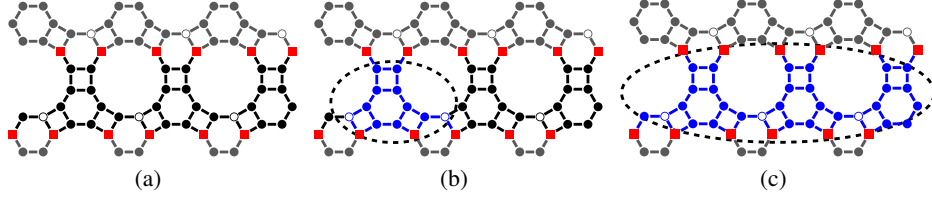


Fig. 10. Threshold-coloring the $(4, 6, 12)$ lattice. (a) Identifying the patches from lemma 4. Observe that there are alternating “rows” separated by square vertices. (b) One patch has been colored, shown inside the oval. (c) Extending the coloring to an entire row.

Proof (Lemma 4). Let l be an edge labeling of G . We consider only the case where $c(v_0) \in \{2, 4\}$ as the other case is symmetric. First, let $c(v_6) = 1$. Using Lemma 2, we color v_5, v_4 , and v_3 so that $c(v_3)$ is in $\{\pm 1, \pm 4\}$. Consider Table 1, where we list valid colors of v_1 in $\{\pm 2, \pm 4\}$ according to edge labeling and $c(v_3)$. An “x” indicates no color can be chosen, but in these cells we multiply $c(v_3)$ by -1 to obtain a color for v_1 , and we multiply $c(v_4), c(v_5)$, and $c(v_6)$ by -1 so that this is consistent. Use Lemma 2 to choose colors for v_2, v_8, v_7, v_9 , and v_{10} so that $v_{10} \in \{\pm 2, \pm 4\}$. \square

Proof (Theorem 2). Let us prove the claim for the $(4,6,12)$ lattice.

Consider the graph G from Lemma 4. We can construct the $(4,6,12)$ lattice by joining together several copies of G . Suppose we have k copies of G , denoted G_1, G_2, \dots, G_k . If v is a vertex corresponding to v_i or u_j in G_l , then we denote v by $v_{i,l}$ or $u_{j,l}$, $1 \leq l \leq k$. Now, construct a row of the lattice from the copies of G by setting $v_{10,l} = v_{0,l+1}$ and $u_{4,l} = u_{0,l+1}$. Lemma 4 allows us to $(9, 2)$ -total-threshold-color

	$l(v_0, v_1) = N$ $l(v_1, v_3) = F$	$l(v_0, v_1) = N$ $l(v_1, v_3) = N$	$l(v_0, v_1) = F$ $l(v_1, v_3) = N$	$l(v_0, v_1) = F$ $l(v_1, v_3) = F$
$c(v_3) = 1$	4	2	x	-4,-2
$c(v_3) = -1$	2,4	x	-2	-4
$c(v_3) = 4$	x	2,4	x	-4,-2
$c(v_3) = -4$	2,4	x	-4,-2	x

Table 1.

any such chain, by fixing the color $v_{0,1}$ to be 2 or 4 (depending on the edge $(v_{0,1}, u_{0,1})$, and then applying the lemma in sequence to G_1, \dots, G_k .

Given two such chains G_1, \dots, G_k and G'_1, \dots, G'_{k-1} , we can stack the second on top of the first. First, if we have a vertex v corresponding to v_i or u_j in the graph G'_l , then we denote v by $v'_{i,l}$. Now, we join the two chains by identifying the vertex $u_{2,l}$ with $u'_{0,l}$ and the vertices $u_{1,l+1}, u_{2,l+1}$ with $u'_{3,l}$ and $u'_{4,l}$, respectively. By repeatedly adding new rows, we can complete the construction of the $(4,6,12)$ lattice; see Fig. 10(a). The rows are connected only by vertices colored 0, so we can color each row to threshold color the entire lattice. \square

Proof (Lemma 5). Let $c(w_i) = c_i$ for $i \in \{1, 2, 3, 4\}$. After possible renaming assume that if $l(e_1) \neq l(e_2)$ then $l(e_1) = N, l(e_2) = F$ and if $l(e_3) \neq l(e_4)$ then $l(e_3) = N, l(e_4) = F$. Define two integers $\lambda_{1,2}$ and $\lambda_{3,4}$ as follows. If $c_1 < c_2$ then $\lambda_{1,2} = c_2 - k - 1$; otherwise $\lambda_{1,2} = c_2 + k + 1$. Similarly, if $c_3 < c_4$ then $\lambda_{3,4} = c_4 - k - 1$; otherwise $\lambda_{3,4} = c_4 + k + 1$. For both values of $\lambda_{i,j}$ we have $|c_i - \lambda_{i,j}| \leq k$; but $|c_j - \lambda_{i,j}| = k + 1$. Also since $c_i \in \{k + 2, \dots, 2k + 1\}$, the value for $\lambda_{i,j}$ is in the set $\{1, \dots, 3k + 2\}$. Table 2 lists the colors $c(u)$ and $c(v)$ assigned to u and v for all possible edge labeling l assuming that the coloring for $V - \{u, v\}$ is extendible with respect to l .

Here by our definition of e_1, e_2, e_3 and e_4 , it is not possible that $l(e_1) = F$ and $l(e_2) = N$, or $l(e_3) = F$ and $l(e_4) = N$. Again for each value of $\lambda_{i,j}; i, j = 1, 2$ or $3, 4$, exactly one of the two values $k + 1$ and $2k + 2$ is within distance k of $\lambda_{i,j}$ and the other value is more than distance k away. The same also holds for the two values 1 and $3k + 2$. Thus depending on the edge-label of e , there is exactly one valid choice for $c(u)$ and $c(v)$ in columns 2, 5, 8 and 14 of Table 2. Finally in column 6, we claim that $\lambda_{1,2}$ and $\lambda_{3,4}$ are within distance k of each other if and only if $l(e_0) = N$. Without loss of generality assume that $c_1 < c_2$. Then $\lambda_{1,2} = c_2 - k - 1 \in \{1, \dots, k\}$ since $c_2 \in \{k + 2, \dots, 2k + 1\}$. Now since c is extendible with respect to l , $c_3 < c_4$ if and only if $l(e) = N$. Thus if $l(e) = N$, $\lambda_{3,4} = c_4 - k - 1 \in \{1, \dots, k\}$ since $c_4 \in \{k + 2, \dots, 2k + 1\}$. Therefore, $|\lambda_{1,2} - \lambda_{3,4}| < k$. On the other hand if $l(e_0) = F$, then $c_3 > c_4$ and hence $\lambda_{3,4} = c_3 + k + 1 \in \{2k + 2, \dots, 3k + 1\}$. Thus $|\lambda_{1,2} - \lambda_{3,4}| > k$. Thus the assignment of $c(u)$ and $c(v)$ respects the edge labeling in all cases. \square

Proof (Theorem 4). We threshold-color $D(3^4, 6)$ with the same strategy as in Theorem 3. Let G be a particular instance of a $D(3^4, 6)$ graph and let l be a particular edge labeling of G . We construct an independent set V_1 of G with all the 6-vertices and some 3-vertices; see Fig. 11(a). The remaining 3-vertices V_2 induce a matching in G . Let m

		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$l(e_1) =$		N	N	N	N	N	N	N	N	F	F	F	F	F	F	F	F
$l(e_2) =$		N	N	N	N	F	F	F	F	N	N	N	N	F	F	F	F
$l(e_3) =$		N	N	F	F	N	N	F	F	N	N	F	F	N	N	F	F
$l(e_4) =$		N	F	N	F	N	F	N	F	N	F	N	F	N	F	N	F
$l(e_0) = N$	$c(u)$	$k+1$	$k+1$ or $2k+2$	-	$k+1$	$\lambda_{1,2}$	$\lambda_{1,2}$	-	$\lambda_{1,2}$	-				1	1 or $3k+2$	-	1
	$c(v)$	$k+1$	$\lambda_{3,4}$		1	$k+1$ or $2k+2$	$\lambda_{3,4}$		1 or $3k+2$					$k+1$	$\lambda_{3,4}$		1
$l(e_0) = F$	$c(u)$	$k+1$	$k+1$ or $2k+2$	-	$k+1$	$\lambda_{1,2}$	$\lambda_{1,2}$	-	$\lambda_{1,2}$	-				$3k+2$	1 or $3k+2$	-	1
	$c(v)$	$2k+2$	$\lambda_{3,4}$		$3k+2$ or $2k+2$	$k+1$	$\lambda_{3,4}$		1 or $3k+2$					$k+1$	$\lambda_{3,4}$		$3k+2$

Table 2. Proof of Lemma 5: Assignment of $c(u)$ and $c(v)$ for all possible edge labeling l of G and coloring c of $V - \{u, v\}$

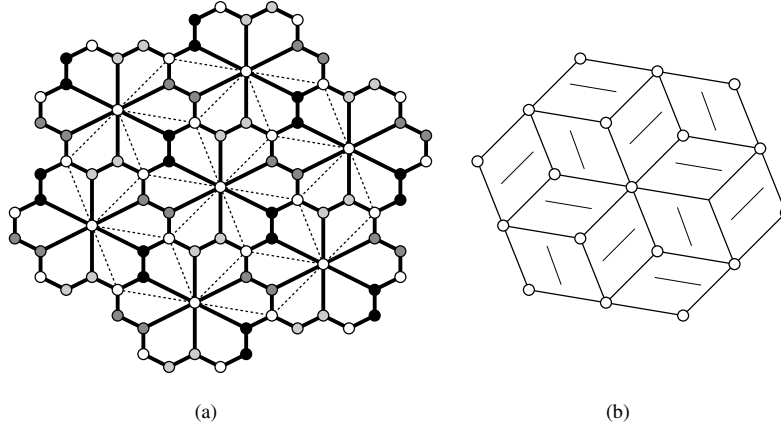


Fig. 11. Illustration for the proof of Theorem 4.

be the number of vertices in V_1 . We first color these vertices with m colors so that each of them gets a unique color. We assign threshold $t = m$. Each edge (u, v) between two vertices from V_2 is surrounded by exactly four vertices from V_1 and they are the other neighbors of u and v . We want to color the vertices of V_1 so that this coloring is extendible to the edge labeling l for all such edges (u, v) , $u, v \in V_2$.

We consider only the vertices of V_1 and add an edge between two vertices of V_1 if and only if they have a common neighbour in G to obtain a $D(3, 6, 3, 6)$ grid H ; see Fig. 11(b). As in Theorem 3, if we direct each edge of H from the vertex with smaller color to the vertex with larger color, the requirement of extendibility of coloring imposes some constraints on the edge directions between either the left and right edges or the top and bottom edges. We now orient all the edges in H so that these constraints

are satisfied and the graph H becomes a directed acyclic graph. We traverse H from left-top to right-bottom and orienting edges so that the constraints are satisfied and the right-bottom vertex for each rectangle in H becomes either a source or sink. This orientation gives a desired directed acyclic graph from H and we color the vertices of V_1 with a topological order of this DAG. We again shift the color space from $\{1, \dots, m\}$ to $\{m+2, \dots, 2m+1\}$ and since this coloring is extendible, by Lemma 5 we have a threshold coloring of G with $3m+2$ colors and a threshold value of m . \square

Proof (Theorem 6). We first prove the claim for the $D(3, 4, 6, 4)$ lattice. Consider the edge labeling of $D(3, 4, 6, 4)$ where the near edges form a spiral-shaped path P ; see Fig. 12(a). Each face except the one with vertices labeled v_1, v_2, v_3, v_4 has exactly 2 near edges. We can add vertices from $D(3, 4, 6, 4)$ to the end of this path, so that the near edges are still a path and each newly added face has exactly 2 near edges. We label the vertices, starting at v_1 , using breadth first search, so that the successor of vertex v_i is always v_{i+1} . Suppose c is a threshold-coloring such that $c(v_1) < c(v_4)$. Now, consider a vertex v_i , $i > 4$, with a neighbour v_j such that $j < i$ and $l(v_i, v_j) = F$. Then v_i, v_j are on a face (v_i, v_{i+1}, v_k, v_j) . There are 2 far edges; they are either (i) (v_i, v_j) and (v_k, v_j) or (ii) (v_i, v_j) and (v_{i+1}, v_k) . In (i), if $c(v_k) > c(v_j)$, then also $c(v_i) > c(v_j)$ by Lemma 6, and in (ii), if $c(v_{i+1}) > c(v_k)$ then $c(v_i) > c(v_j)$, by the same lemma. Since $c(v_4) > c(v_1)$, we can inductively conclude that $c(v_i) > c(v_j)$, and hence that as we add vertices to the path P , we require more and more colors.

We now prove the claim for the $D(3, 6, 3, 6)$ lattice. Consider the edge labeling of the $D(3, 6, 3, 6)$ lattice shown in Fig. 12(b), with vertex set V . The set of near edges forms a spiral-shaped caterpillar T . All but one interior face has exactly 2 near edges. We extend this caterpillar by adding vertices to the end of T , and labeling the edges so that every interior face has exactly 2 near edges, except the one mentioned earlier. Let l be this edge labeling. Now, suppose c is a threshold coloring of the caterpillar, and without loss of generality choose the colors of v_1, v_4 in Fig. 12(b) so that $c(v_1) < c(v_4)$. We can traverse the vertices of T using breadth first search. Whenever we traverse a vertex with degree 3 in T , we traverse its degree 1 neighbour before we traverse its higher degree neighbors. Assume that v is a vertex with a neighbour u traversed prior to v , such that $l(u, v) = F$ and $c(v) > c(u)$. If v is not a leaf, then v is on a face (x, y, v, u) either with (i) x, y not traversed, and $l(x, u) = F$, or (ii) y not traversed, and $l(y, x) = F$. Since $l(u, v) = F$, the other edges are near. Then by Lemma 6, we have in (i) that $c(x) > c(u)$ and in (ii) that $c(y) > c(x)$ (both since $c(v) > c(u)$). Now, since $c(v_4) > c(v_1)$, we can traverse T and conclude that each vertex y with a previously traversed vertex x such that $l(x, y) = F$ must have $c(y) > c(x)$. Therefore, for any $r > 0$, as the caterpillar T grows arbitrarily big, we require more than r colors to threshold-color it. \square

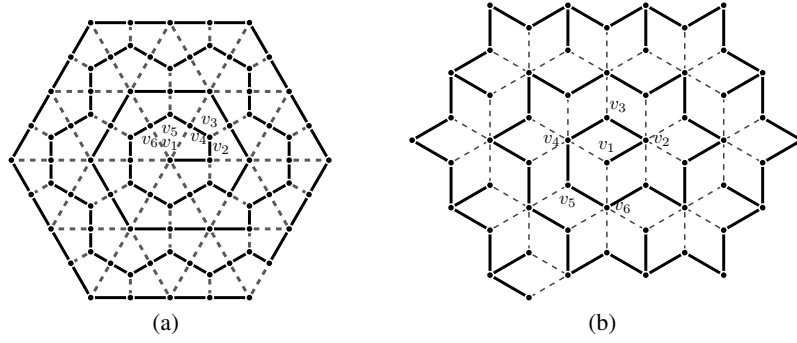
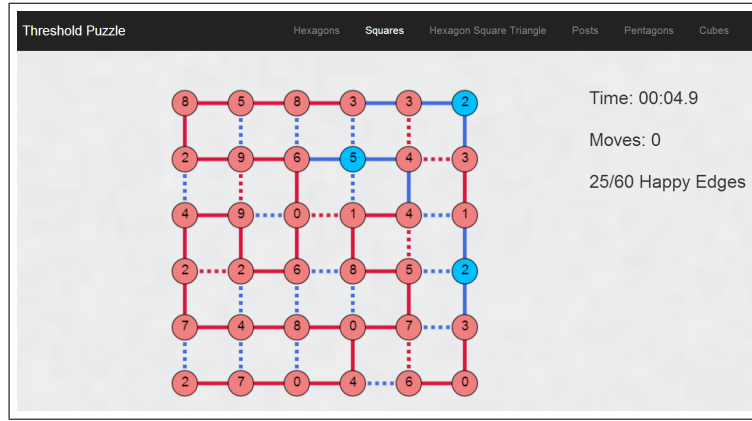
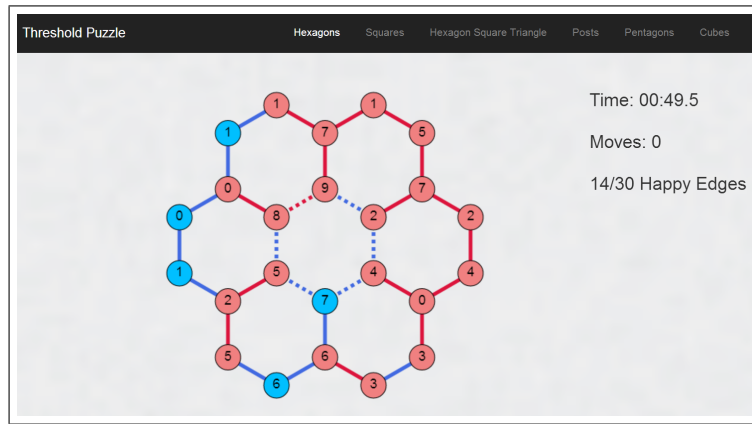


Fig. 12. Subgraphs of the (a) $D(3,4,6,4)$ lattice and (b) $D(3,6,3,6)$ lattice, which require arbitrarily many colors. Dashed edges are labeled F .



(a)



(b)

Fig. 13. Snapshot of the *Happy Edges* puzzle environment available for computers and mobile devices. Dashed edges indicate F edges, and vertices/edges are blue if the color of their neighbors/endpoints are correct. (a) The square lattice. (b) The triangle square hexagon lattice.